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Hydromagnetic source flow

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The steady flow of a viscous incompressible and electrically conducting fluid between two parallel coaxial rotating disks with a transverse magnetic field has been analysed when the two disks are rotating with the same velocity and there is a source at the centre. The velocity of the fluid relative to the disks is supposed to be small

1. Introduction

The steady laminar source flow between two parallel disks rotating in the same direction with the same constant angular velocity has been studied by Kreith & Peube (1965, 1966) and Breiter & Pohlhausen (1962) Recently Khan (1968) studied the effect of porosity on the laminar source flow between two disks rotating with the same velocity. Further Khan (1969) extended the problem of Kreith & Peube to hydromagnetics and calculated the effect of uniform transverse magnetic field for the two cases, *viz.*,

(*i*) when the applied transverse magnetic field is small,

and (*ii*) when the rotational Taylor numbers of the two disks are small

In the present paper an attempt has been made to analyse the flow of a viscous, incompressible and electrically conducting fluid in presence of a constant transverse magnetic field, when there is no restriction on the strength of the magnetic field (but the induced field, is neglected), or on the rotational Taylor numbers of the two disks, but it has been supposed that the velocity of the fluid relative to the velocities of the disks is small Source flow problems of this type are of interest in the design of viscosity pumps (Hasinger & Kehrt 1963, Rice 1963), rotating heat exchangers (Clark & Bromley 1961), air thrust bearings (Grassam & Powell 1964), multiple disk turbines (Armstrong 1952, Gurbert 1960), in calculating the loss of lift in ground effect on VOLT aircraft with centrally located downward pointing jet, and in the design of radial diffusers, particularly with reference to the design of efficient internal ducts in annular jet ground effect machines

2. Basic equations and boundary conditions

Let us take the axis of rotation of the two disks as z-axis and let the two disks be situated at $z = \pm a$. Consider the flow of an incompressible fluid between two parallel rotating disks with the source at the centre.

The governing non-dimensional hydromagnetic equations of motion in cylindrical polar coordinates are

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r} = -\frac{\partial p}{\partial r}+\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+\frac{\partial^2 u}{\partial z^2}-Mu, \tag{2.1}$$

$$u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r} = \frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)+\frac{\partial^2 v}{\partial z^2}-Mr, \tag{2.2}$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z}+\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}, \tag{2.3}$$

and the equation of continuity

$$\frac{\partial}{\partial r}(ru)+\frac{\partial}{\partial z}(rw) = 0 \tag{2.4}$$

where the non-dimensional quantities are defined as

$$r = \frac{\bar{r}}{\sqrt{\nu/\omega}}, \quad z = \frac{\bar{z}}{\sqrt{\nu/\omega}}, \quad u = \frac{\bar{u}}{\sqrt{\nu\omega}},$$

$$v = \frac{\bar{v}}{\sqrt{\nu\omega}}, \quad w = \frac{\bar{w}}{\sqrt{\nu\omega}}, \quad p = \frac{\bar{p}}{\rho\nu\omega}, \quad M = \frac{\sigma B_0^2}{\rho\omega}$$

If the two disks rotate with angular velocity ω and the strength of the source is Q, the boundary conditions are

$$\left.\begin{aligned} u &= 0 \\ v &= r \\ w &= 0 \end{aligned}\right\} \text{ at } z = \pm\alpha \tag{2.5}$$

$$r\int_{-\alpha}^{+\alpha} u\,dz = K. \tag{2.6}$$

where $\alpha = \dfrac{a}{\sqrt{\nu/\omega}}$ and $K = \dfrac{Q\omega^{\frac{1}{2}}}{2\pi\nu^{3/2}}$

3 Solution of the problem

It is convenient to introduce instead of v, which is the non-dimensional tangential velocity in the fixed system of coordinates, the non-dimensional tangential velocity relative to the disks

$$V = v - r.$$

Substituting $v = V+r$ in the equations of motion and assuming that the velocity of fluid relative to the disks is small (this condition is well statisfied for large radii), so that the quadratic terms are negligible, we obtain

$$-2V - r = -\frac{\partial p}{\partial r} + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2} - Mu, \tag{3.1}$$

$$2u = \frac{\partial}{\partial r}\left(\frac{\partial V}{\partial r} + \frac{V}{r}\right) + \frac{\partial^2 V}{\partial z^2} - M(V+r), \tag{3.2}$$

$$0 = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2} \tag{3.3}$$

and the no-slip boundary conditions then become

$$\left.\begin{aligned} u(r, \pm\alpha) &= 0, \\ v(r, \pm\alpha) &= 0, \\ w = (r, \pm\alpha) &= 0 \end{aligned}\right\} \tag{3.4}$$

The equation of continuity (2.4) is satisfied by introducing a stream function ψ defined by

$$u = \frac{1}{r}\frac{\partial \psi}{\partial z}, \quad w = -\frac{1}{r}\frac{\partial \psi}{\partial r} \tag{3.5}$$

The solution can then be expressed in the form

$$u = rf'_{-1} + f'_0 + \frac{f'_1}{r} + \frac{f'_2}{r^2} + \frac{f'_3}{r^3} + \ldots + \frac{f'_n}{r} + \ldots, \tag{3.6}$$

$$V = rg_{-1} + g_0 + \frac{g_1}{r} + \frac{g_2}{r^2} + \frac{g_3}{r^3} + \ldots + \frac{g_n}{r} + \ldots, \tag{3.7}$$

$$w = -2f_{-1} - \frac{f_0}{r} + \frac{f_2}{r^3} + \ldots + \frac{(n-1)f_n}{r^{n+1}} + \ldots, \tag{3.8}$$

$$p = r^2 h_{-2} + rh_{-1} + h_0 + h_1 \log r + \frac{h_2}{r} + \frac{h_3}{r^2} + \ldots + \frac{h_n}{r^{n-1}} + \ldots, \tag{3.9}$$

Substituting these values of u, V, w and p in equations (3 1) to (3 3) and equating terms in like powers of r on both sides, we get an infinite system of simultaneous ordinary differential equations. The first five systems are

$$\text{(I)} \qquad \begin{cases} f'''_{-1}+2g_{-1}+1 = 2h_{-2}+Mf'_{-1}, \\ g''_{-1}-2f'_{-1} = M(g_{-1}+1), \\ h'_{-2} = 0. \end{cases}$$

$$\text{(II)} \qquad \begin{cases} f'''_{0}+2g_{0} = h_{-1}+Mf'_{0}, \\ g''_{0}-2f'_{0} = Mg_{0}, \\ h'_{-1} = 0. \end{cases}$$

$$\text{(III)} \qquad \begin{cases} f'''_{1}+2g_{1} = h_{1}+Mf'_{1}, \\ g''_{1}-2f'_{1} = Mg_{1}, \\ h'_{1} = 0. \end{cases}$$

$$\text{(IV)} \qquad \begin{cases} f'''_{2}+2g_{2} = -h_{2}+f'_{0}+Mf'_{2}, \\ g''_{2}+2f'_{2} = g_{0}+Mg_{2} \\ h'_{2} = f''_{0} \end{cases}$$

$$\text{(V)} \qquad \begin{cases} f'''_{3}+2g_{3} = 2h_{3}+Mf'_{3}, \\ g''_{3}-2f'_{3} = Mg_{3}, \\ h'_{3} = 0 \end{cases}$$

Solving system (I) and applying boundary conditions, we have

$$\begin{aligned} f'_{-1}(z) &= A_1 \cosh (z\sqrt{r}\cos\theta/2)\cos(z\sqrt{r}\sin\theta/2) \\ &\quad + B_1 \sinh (z\sqrt{r}\cosh\theta/2)\sin(z\sqrt{r}\sin\theta/2) - \frac{2Mh_{-2}+M}{M^2+4}, \end{aligned}$$

$$\begin{aligned} g_{-1}(z) &= A_1 \sinh (z\sqrt{r}\cos\theta/2)\sin(z\sqrt{r}\sin\theta/2) \\ &\quad - B_1 \cosh (z\sqrt{r}\cos\theta/2)\cos(z\sqrt{r}\sin\theta/2) + \frac{4h_{-2}-(M^2+2)}{M^2+4}, \end{aligned}$$

where $r = \sqrt{M^2+4}$, $\theta = \tan^{-1} 2/M$ and A_1, B_1 and h_{-2} are constants given by

$$A_1 = 2\left[\frac{2Mh_{-2}+M}{M^2+4}\right][\cos\theta/2\cosh(\alpha\sqrt{r}\cos\theta/2)\sin(\alpha\sqrt{r}\sin\theta/2)$$

$$-\sin\theta/2\sinh(\alpha\sqrt{r}\cos\theta/2)\cos(\alpha\sqrt{r}\sin\theta/2)$$

$$-\alpha\sqrt{r}\sinh(\alpha\sqrt{r}\cos\theta/2)\sin(\alpha\sqrt{r}\sin\theta/2)]$$

$$\div[\cos\theta/2\sin(2\alpha\sqrt{r}\sin\theta/2)-\sin\theta/2\sinh(2\alpha\sqrt{r}\cos\theta/2)],$$

$$B_1 = -2\left[\frac{2Mh_{-2}+M}{M^2+4}\right][\sin\theta/2\cosh(\alpha\sqrt{r}\cos\theta/2)\sin(\alpha\sqrt{r}\sin\theta/2)$$

$$+\cos\theta/2\sinh(\alpha\sqrt{r}\cos\theta/2)\cos(\alpha\sqrt{r}\sin\theta/2)$$

$$-\alpha\sqrt{r}\cosh(\alpha\sqrt{r}\cos\theta/2)\cos(\alpha\sqrt{r}\sin\theta/2)]$$

$$\div[\cos\theta/2\sin(2\alpha\sqrt{r}\sin\theta/2)-\sin\theta/2\sinh(2\alpha\sqrt{r}\cos\theta/2)],$$

$$h_{-2} = \left\{\frac{2M^2+4}{M^2+4}-\frac{2M}{M^2+4}\left[\frac{\cos\theta/2\sinh(2\alpha\sqrt{r}\cos\theta/2)+\sin\theta/2\sin(2\alpha\sqrt{r}\sin\theta/2)-\alpha\sqrt{r}\{\cosh(2\alpha\sqrt{r}\cos\theta/2)+\cos(2\alpha\sqrt{r}\sin\theta/2)\}}{\cos\theta/2\sin(2\alpha\sqrt{r}\sin\theta/2)-\sin\theta/2\sinh(2\alpha\sqrt{r}\cos\theta/2)}\right]\right\}$$

$$\div\left\{\frac{8}{M^2+4}+\frac{4M}{M^2+4}\left[\frac{\cos\theta/2\sinh(2\alpha\sqrt{r}\cos\theta/2)+\sin\theta/2\sin(2\alpha\sqrt{r}\sin\theta/2)-\alpha\sqrt{r}\{\cosh(2\alpha\sqrt{r}\cos\theta/2)+\cos(2\alpha\sqrt{r}\sin\theta/2)\}}{\cos\theta/2\sin(2\alpha\sqrt{r}\sin\theta/2)-\sin\theta/2\sinh(2\alpha\sqrt{r}\cos\theta/2)}\right]\right\}$$

Putting $M = 0$, the values of $f'_{-1}(z)$ $g_{-1}(z)$ and h_{-2} reduce to those obtained by Kreith & Peube (1965)

Solving the system (II) we have

$$f_0 = g_0 = h_{-1} = 0$$

On solving system (III) we have

$$f'_1(z) = A_2\cosh(z\sqrt{r}\cos\theta/2)\cos(z\sqrt{r}\sin\theta/2)$$
$$+B_2\sinh(z\sqrt{r}\cos\theta/2)\sin(z\sqrt{r}\sin\theta/2)-\frac{Mh_1}{M^2+4},$$

$$g_1(z) = A_2\sinh(z\sqrt{r}\cos\theta/2)\sin(z\sqrt{r}\sin\theta/2)$$
$$-B_2\cosh(z\sqrt{r}\cos\theta/2)\cos(z\sqrt{r}\sin\theta/2)+\frac{2h_1}{M^2+4},$$

where

$$A_2 = \left[\frac{2Mh_1}{M^2+4}\right] [\cos\theta/2 \cosh(\alpha\sqrt{r}\cos\theta/2)\sin(\alpha\sqrt{r}\sin\theta/2)$$

$$-\sin\theta/2 \sinh(\alpha\sqrt{r}\cos\theta/2)\cos(\alpha\sqrt{r}\sin\theta/2)$$

$$-\left\{\alpha\sqrt{r}+\frac{K\sqrt{r}}{2}\left(\frac{M^2+4}{Mh_1}\right)\right\}\sinh(\alpha\sqrt{r}\cos\theta/2)\sin(\alpha\sqrt{r}\sin\theta/2)]$$

$$-[\cos\theta/2\sin(2\alpha\sqrt{r}\sin\theta/2)-\sin\theta/2\sinh(2\alpha\sqrt{r}\cos\theta/2)],$$

$$B_2 = -\left[\frac{2Mh_1}{M^2+4}\right] [\sin\theta/2\cosh(\alpha\sqrt{r}\cos\theta/2)\sin(\alpha\sqrt{r}\sin\theta/2)$$

$$+\cos\theta/2\sinh(\alpha\sqrt{r}\cos\theta/2)\cos(\alpha\sqrt{r}\sin\theta/2)$$

$$-\left\{\alpha\sqrt{r}+\frac{K\sqrt{r}}{2}\left(\frac{M^2+4}{Mh_1}\right)\right\}\cosh(\alpha\sqrt{r}\cos\theta/2)\cos(\alpha\sqrt{r}\sin\theta/2)]$$

$$+[\cos\theta/2\sin(2\alpha\sqrt{r}\sin\theta/2)-\sin\theta/2\sinh(2\alpha\sqrt{r}\cos\theta/2)],$$

$$h_1 = K\sqrt{r}\{\cosh(2\alpha\sqrt{r}\cos\theta/2)+\cos(2\alpha\sqrt{r}\sin\theta/2)\}$$

$$\div\left[\frac{4}{M^2+4}\{\cos\theta/2\sin(2\alpha\sqrt{r}\sin\theta/2)-\sin\theta/2\sinh(2\alpha\sqrt{r}\cos\theta/2)\}\right.$$

$$+\frac{2M}{M^2+4}\{\cos\theta/2\sinh(2\alpha\sqrt{r}\cos\theta/2)+\sin\theta/2\sin(2\alpha\sqrt{r}\sin\theta/2)$$

$$\left.-\alpha\sqrt{r}(\cosh(2\alpha\sqrt{r}\cos\theta/2)+\cos(2\alpha\sqrt{r}\sin\theta/2))\}\right].$$

Putting $M=0$, the values of $f'_1(z)$, $g_1(z)$ and h_1 again reduce to those obtained by Kreith & Peube (1965)

On solving other systems we have

$$f'_n = g_n = 0, \qquad (n = 2, 3, \ldots)$$

Table 1. $\alpha = 1$

Z \ $1/Kf'_1(z)$	$M = 0$	$M = 1$	$M = 10$
0.0	0.7278	0.7236	0.6658
0.1	0.7216	0.7177	0.6628
0.2	0.7031	0.6997	0.6535
0.3	0.6716	0.6695	0.6369
0.4	0.6265	0.6259	0.6111
0.5	0.5669	0.5678	0.5736
0.6	0.4915	0.4941	0.5203
0.7	0.3988	0.4026	0.4464
0.8	0.2872	0.2916	0.3414
0.9	0.1549	0.1584	0.1972
1.0	0.0000	0.0000	0.0000

4. Discussion

Curves have been drawn showing the variation of $f'_{-1}(z)$, $g_{-1}(z)$ and $g_1(z)$ with z when $\alpha = 1$ and $M = 0$, 1 and 10. The variation of $f'_1(z)$ with z is shown in table 1.

From figure 1, it is seen that for the magnetic case ($M = 1$) the value of $f'_{-1}(z)$ is zero at the two disks. It increases rapidly as the non-dimensional

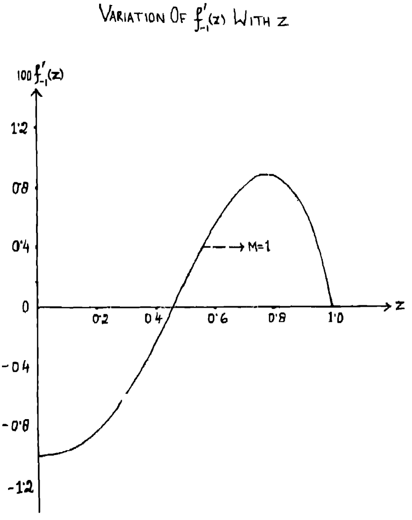

Figure 1

distance from the disks increases from 0 to $\pm 0{\cdot}22$ It attains its greatest value at $z \simeq 0{\cdot}78$ The value then decreases rapidly and again attains zero value at $z \simeq \pm 0{\cdot}45$ It then becomes negative and its numerical value increases in the same manner till it becomes almost constant in the small region midway between the two disks The curve showing the variation of $f'_{-1}(z)$ with z is of an oscillatory nature and is symmetrical about $z = 0$.

For the non-magnetic case *i.e.*, when $M = 0$, $f'_{-1}(z)$ is zero throughout the region between the two disks We, therefore, infer that the magnetic field increases the value of $f'_{-1}(z)$ in the regions bounded by $z \simeq (0.45, 1)$ and $z \simeq (-0{\cdot}45, -1)$ while it damps the value of $f'_{-1}(z)$ in the region $z \simeq (0.45, -0{\cdot}45)$.

Again from table 1 it is observed that the variation of $f'_1(z)$ with z for the magnetic and non-magnetic cases is almost the same *i.e.*, the effect of magnetic field on the value of $f'_1(z)$ is very small The magnetic field increases and decreases the value of $f'_1(z)$ in the same regions as it does in the case of $f'_{-1}(z)$.

From the above discussion we infer that when $M = 1$. the effect of magnetic field is to increase the radial velocity in the regions bounded by $z \simeq (0.45, 1)$ and $z \simeq (-0{\cdot}45, -1)$. while it damps it in the region bounded by $z \simeq (0{\cdot}45. -0{\cdot}45)$.

VARIATION OF - $g_{-1}(z)$ WITH z

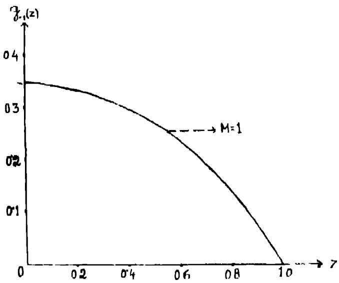

Figure 2

VARIATION OF $g_1(z)$ WITH z

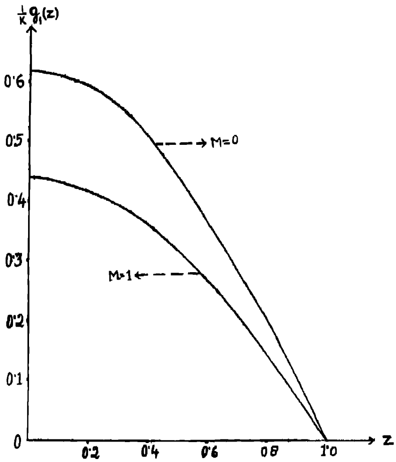

Figure 3

From figures 2 and 3 we observe that the pattern of variation of $g_{-1}(z)$ and $g_1(z)$ with z is almost the same. It is seen that the transverse velocity V relative to the disks is always negative and its amplitude is maximum in the region midway between the two disks It decreases to zero as z increases from $z = 0$ to $z = \pm 1$ It is obvious from the two figures that the effect of magnetic field is to damp the transverse velocity

Hence we see that when $M = 1$, the effect of magnetic field is similar to that found by Khan (1969) when the transverse magnetic field is small

Again from figure 4 we see that when $M = 10$, the effect of magnetic field is to damp the radial as well as the transverse velocity throughout the region between the two disks

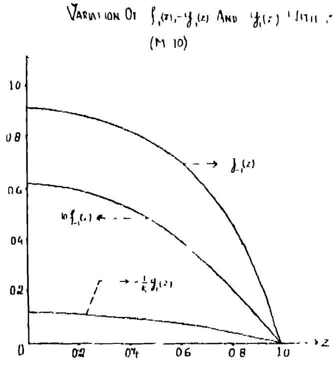

Figure 4

Hence we may conclude that for small and moderate values of the magnetic parameter M, the effect of magnetic field is to increase the radial velocity in the region near the two disks, while it damps it in the region midway between the two disks However, for large values of M the effect of magnetic field is to damp the radial velocity for every region between the two disks. The effect of the magnetic field on transverse velocity is to damp it for all values of the magnetic parameter M The damping effect is large for large values of M and small for small values of the magnetic parameter

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